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Optimization in normed vector spaces with applications to optimal economic growth theory

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1. The structure of optimal economic growth and investment models.

In many economical growth and investment models the set of growth paths, which are feasible in economic-technical respects, can be characterized by a system of inequalities:

$$\left.\begin{array}{r} B[x(t+1);t+1]-A[x(t);t] \leqq f(t+1) \\ x(t) \geqq 0 \end{array}\right\} \quad t = 0,1,\ldots, \qquad (1.1)$$

where the growth path is represented by the sequence of n-dimensional vectors $\{x(t)\}_0^\infty$ and where the structure of the model is given by the functions $B[\cdot;t] : R_+^n \to R^m$, $A[\cdot;t] : R_+^n \to R^m$ and by the sequence of m-dimensional right-hand vectors $\{f(t)\}_1^\infty$.

The following conditions are often imposed:

(a) the functions $B[\cdot;t]$ are convex, the functions $A[\cdot;t]$ are concave, moreover $B[0;t] = 0$, $A[0;t] = 0$

(b) for every $x \in R_+^n$, the sequences $\{A[x;t]\}_0^\infty$ and $\{B[x;t]\}_1^\infty$ are bounded. The functions are all continuous.

Paths $\{x(t)\}_0^\infty$ which satisfy the restrictions (1.1), further to be called <u>feasible solutions</u> or <u>feasible paths</u>, will be compaired mutually with respect to a sequence of objective functions:

$$\{\sum_{t=1}^{T} \pi^t p[x(t);t]\}_{T=1}^\infty \qquad (1.2)$$

where the functions $p[\cdot;t] : R_+^n \to R^1$ are continuous, and such that for every $x \in R_+^n$ the sequence of numbers $\{p[x;t]\}_{t=1}^\infty$ is bounded. Moreover, $p[0;t] = 0$, $t = 1,2,\ldots$. The coefficient $\pi \in]0,1[$ gives the weight of a succeeding period with respect to its preceding period.

Given the initial vector $x(0) \in R_+^n$, we call a path $\{\hat{x}(t)\}_1^\infty$ <u>superior</u> with respect to $x(0)$, if it is feasible and if, in addition, for the same initial vector $x(0)$ no feasible solution $\{\bar{x}(t)\}_1^\infty$ exists, such that for some $\varepsilon > 0$ and some integer $S \geqq 1$:

$$\sum_{t=1}^{T} \pi^t p[\bar{x}(t);t] \geqq \varepsilon + \sum_{t=1}^{T} \pi^t p[\hat{x}(t);t], \quad T = S, S+1, \ldots \tag{1.3}$$

Clearly, in this manner, the maximizing of an objective functions is replaced by a process of mutually comparization of feasible solutions with respect to a sequence of objective functions. In this context, we call a subset $\underset{\sim}{F}$ of feasible solutions a pre-superior set, if, for every feasible path $\{\hat{x}(t)\} \notin \underset{\sim}{F}$, a path $\{\bar{x}(t)\}_1^\infty \in \underset{\sim}{F}$ exists satisfying (1.3) for some $\varepsilon > 0$ and some $S \geqq 1$.

Now, the programming problem can be formulated as the search for superior solutions and for pre-superior sets. In many cases it is possible to give pre-superior sets, which are situated in a normed space for which the sequences of functions $\{B[\cdot;t]\}_1^\infty$, $\{A[\cdot;t]\}_1^\infty$, $\{p[\cdot;t]\}_1^\infty$ have nice properties, for instance conditions which imply the convergence of the series (1.2).
In that case, only the limit

$$\sum_{t=1}^{\infty} \pi^t p[x(t);t] := \lim_{T\to\infty} \sum_{t=1}^{T} \pi^t p[x(t);t], \tag{1.4}$$

is important, and we prefer to speak of optimal solutions, rather than of superior solutions.

In this study we deduce some conditions implying the existence of optimal solutions and which allow us to develope a duality theory. This means that these results are applicable in those cases in which pre-superior sets can be found fitting the conditions mentioned above.

2. Abstract formulation of the optimal growth problem.

In the abstract treatment of the problem, we shall use some spaces which are congruent (viz. ref. 4, page 84) with the l_1-space or the l_∞-space.
Therefore we define:
- The space of sequences of k-dimensional vectors:

$$l^k := \{x := \{x(t)\}_1^\infty \mid x(t) \in R^k,\ t = 1,2,\ldots\} \tag{2.1}$$

- The following transformation of l_1-spaces:

$$l^k_{1;\alpha} := \{\{x(t)\}_1^\infty \in l^k \mid \sum_{t=1}^{\infty} \sum_{i=1}^{k} |\alpha^t x(t)_i| < \infty\} \tag{2.2}$$

- The following transformation of l_∞-spaces:

$$l^k_{\infty;\alpha} := \{\{x(t)\}_1^\infty \in l^k \mid \sup_t \max_i |\alpha^t x_i(t)| < \infty\} \tag{2.3}$$

The conditions appearing on the right-hand side of (2.2) and (2.3) are the norms of $l^k_{1;\alpha}$ and of $l^k_{\infty;\alpha}$ resp. Clearly, if $\alpha > 0$ there are one-to-one correspondences between l_1 and $l^k_{1;\alpha}$, and between l_∞ and $l^k_{\infty;\alpha}$.

We define the positive cone of l^k, $l^k_{1;\alpha}$ and $l^k_{\infty;\alpha}$ by:
$l^k_+ := \{\{x(t)\}_1^\infty \in l^k \mid x(t) \in R^k_+,\ t = 1,2,\ldots\}$,
$l^k_{1;\alpha+} := l^k_{1;\alpha} \cap l^k_+$, $l^k_{\infty;\alpha+} := l^k_{\infty;\alpha} \cap l^k_+$.

Now, starting from the functions $A[\cdot;t]$, $B[\cdot;t]$, and $p[\cdot;t]$, appearing in (1.1) and (1.2), we define the functions $G\colon l^n_{\infty;1+} \to l^m_{\infty;1}$ and $q := l^n_{\infty;1+} \to R^1$ in such a manner that for every $x := \{x(t)\}_1^\infty \in l^n_{\infty;1}$:

$$G[x] = \{B[x(1);1],\ -A[x(1);1] + B[x(2);2], \ldots$$

$$\ldots,\ -A[x(t);t] + B[x(t+1);t+1], \ldots\ldots\}. \tag{2.4}$$

$$q[x] := \sum_{t=1}^{\infty} \pi^t p[x(t);t], \tag{2.5}$$

where the existence of (2.5) is ensured by the assumptions concerning π and the function $p[\cdot;t]$ (viz. §1).

Thus, we obtain the following programming problem:

$$\sup_x q[x] \;\middle|\; \begin{array}{l} G[x] \leqq g \\ x \in l^n_{\infty;1+} \end{array}, \tag{2.6}$$

where $g' := \{f(1)'+A[x(0);0]', f(2)', f(3)', \ldots, f(t)', \ldots\}$, $\{f(t)\}_1^\infty$ being the sequence of right-hand vectors of (1.1) and $x(0) \in R^n_+$ the given initial state vector. In the next paragraphs, we shall analyse first the more general problem: $\{\sup q[x] \mid G[x] \leqq g,\ x \in X_+\}$, X being a normed vector space. The results shall then be applied to (2.6).

3. Definition of the abstract optimization problem.

We study the convex programming problem:

$$\hat{\phi} := \sup_x q[x] \;\middle|\; \begin{array}{l} G[x] \leqq g \\ x \in X_+, \end{array} \tag{3.1}$$

where $G[\cdot] : X_+ \to Z$ is a function on a closed positive cone of a normed vector space X into a normed vector space Z with a closed positive cone Z_+, where $g \in Z$ and where $q[\cdot] : X_+ \to R^1$.
We always presume:

(a) $G[\cdot]$ is convex, $q[\cdot]$ is concave

(b) $G[\cdot]$ and $q[\cdot]$ are bounded

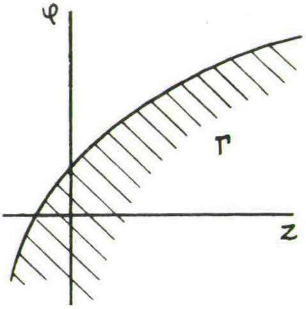
(c) $G[\cdot]$ and $q[\cdot]$ are continuous

(d) $G[0] = 0$ and $q[0] = 0$

A vector $\hat{x} \in X$ is called a feasible solution of (3.1) if: $G[\hat{x}] \leqq g$ and $\hat{x} \in X_+$). A vector $\hat{x} \in X$ is called an optimal solution of (3.1) if it is feasible and if, in addition, $q[\hat{x}] = \hat{\phi}$.

With the help of a set $\Gamma \subset R^1 \times Z$ defined by:

$\Gamma := \{(\phi,z) \in R^1 \times Z \mid \ x \in X_+ : q[x] \geqq \phi, G[x] \leqq g+z\}$,



the following programming problem is joined to (3.1):

$\tilde{\phi} := \sup \ \phi \mid (\phi,0) \in \Gamma.$ (3.2).

The following properties can be easily verified.

4. Proposition.

a) Γ is convex. (By assumption 3-a and def. of Γ).

b) If $(\bar{\phi},\bar{z}) \in \Gamma$, then for every $z \geqq \bar{z}$: $(\bar{\phi},z) \in \Gamma$.

c) $(R^1 \times \{0\}) \subset R^1 \times Z$ is closed.

d) Problem (3.1) possesses a feasible solution if, and only if, $\Gamma \cap (R^1 \times \{0\}) \neq \emptyset$.

e) The supremum in (3.1) is equal to the supremum in (3.2).

f) Problem (3.2) possesses on optimal solution if, and only if, the set $\Gamma \cap (R^1 \times \{0\})$ is non-empty and closed, and if, in addition, the supremum in (3.2) is bounded.

In many cases, it is very hard to deduce usefull properties of the subset $\Gamma_- := \Gamma \cap (R^1 \times Z_-)$. Therefore, the following property can be important.

5. Proposition.

Suppose $\Gamma \cap (R^1 \times \{0\})$ is non-empty, and suppose that the supremum ϕ in (3.2) is bounded. Then, closedness of $\Gamma_+ := \Gamma \cap (R^1 \times Z_+)$ implies:

a) For every positive number $\varepsilon > 0$: $(\tilde{\phi}+\varepsilon, 0) \notin \bar{\Gamma}$.

b) Problem (3.2) possesses an optimal solution, and so, by virtue of 4-f, problem (3.1) as well.

(Note: the closure of a set S is denoted by $\bar{S}$).

Proof: The conditions appearing in this proposition imply:
$[-\infty,\tilde{\phi}] \times \{0\} = \overline{\Gamma_+ \cap (R^1 \times \{0\})} = \overline{\bar{\Gamma}_+ \cap (R^1 \times \{0\})} = \bar{\Gamma}_+ \cap (R^1 \times \{0\})$.

Property 4-b implies: $\bar{\Gamma}_- \subset [-\infty,\tilde{\phi}] \times Z_-$, and so: $\bar{\Gamma}_- \cap (R^1 \times \{0\}) \subset \bar{\Gamma}_+ \cap (R^1 \times \{0\})$, as well. Combining these relations, we may conclude: $[-\infty,\tilde{\phi}] \times \{0\} = \bar{\Gamma} \cap (R^1 \times \{0\})$, which implies the a-part of this proposition.

The b-part can be deduced as follows: $\Gamma \cap (R^1 \times \{0\}) = \Gamma_+ \cap (R^1 \times \{0\}) = \bar{\Gamma}_+ \cap (R^1 \times \{0\}) = \overline{\Gamma_+ \cap (R^1 \times \{0\})}$. This means that $\Gamma \cap (R^1 \times \{0\})$ is closed. Since by assumption $\Gamma \cap (R^1 \times \{0\}) \neq \emptyset$ and $\tilde{\phi}$ is bounded, the closedness of $\Gamma \cap (R^1 \times \{0\})$ implies, by property 4-f, the existence of an optimal solution for (3.2).

6. Example.

The meaning of the conditions of proposition 5, particularly the closedness of $\Gamma_+ := \Gamma \cap (R^1 \times Z_+)$, can be illustrated with the help of the following programming problem:

$$\sup \sum_{i=1}^{\infty} x_i \,\Big|\, \Big(\sum_{i=1}^{\infty} x_i \leqq 1+z_1\Big),\ \Big(\sum_{i=1}^{\infty} \left(\tfrac{1}{2}\right)^i x_i \leqq 0+z_2\Big),\ (x_i \geqq 0,\ i=1,2,\dots)$$

Putting: $X := l_1^1$, $X_+ := \{\{x_i\}_1^\infty \in l_1^1 \mid x_i \geqq 0,\ i = 1,2,\dots\}$, $Z := R^2$, $Z_+ := R_+^2$, this example clearly is as a particular case of problem (3.1).

Putting $(z_1, z_2) := (0,\delta)$, it appears:

a) there is no feasible solution for $\delta < 0$

b) for $\delta = 0$: $x_i = 0$, $i = 1,2,\ldots$ is the only feasible solution, which implies that the supremum is zero

c) for every $\delta > 0$ the supremum is equal to 1.

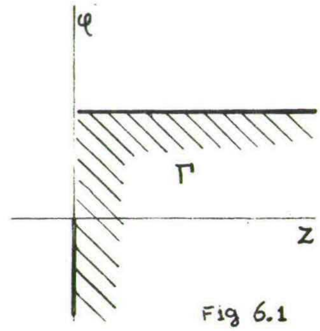


Fig 6.1

Figure 6.1 gives the set $\Gamma_2 := \Gamma \cap \{(\phi, z_1, z_2) \in R^3 | z_1 = 0\}$. Obviously, for this problem the set Γ_+ is not closed, and statement 5-a is not applicable, indeed.

7. Dual problem.

The set of bounded linear functionals $f(x)$ on a normed space X can, by introducing the vector sum and scalar multiplication, be taken as a vector space. If to this space the norm:

$$\| f(\cdot) \|_{X^*} := \sup_x |f(x)| \quad |x \in X, \ \|x\|_X \leqq 1, \tag{7.1}$$

is joined ($\|x\|_X$ being the norm of the space X under consideration), then this vector space is called the (normed) dual space of X.

The common notation is X^* (for more details see for instance Luenberger, ref. 3); bounded linear functionals $f[\cdot] \in X^*$ will be denoted by $\langle f,x \rangle$.

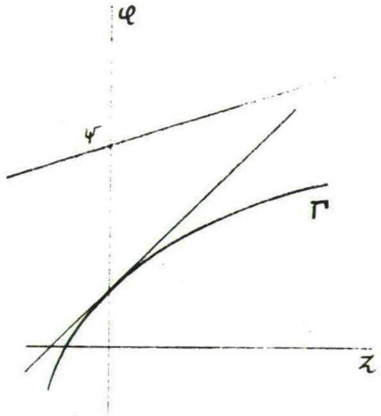
Now, we join the following programming problem to (3.2):

$$\tilde{\psi} := \inf_{(\psi,u^*)} \psi \left| \begin{array}{l} (\psi,u^*) \in R^1 \times Z^* \\ \langle u^*,z \rangle - \phi + \psi \geqq 0, \text{ for all } (\phi,z) \in \Gamma, \end{array} \right. \tag{7.2}$$

$(\psi,u^*) \in R^1 \times Z^*$ is called a feasible solution of (7.2) if the conditions of (7.2) are met. $(\psi,u^*) \in R^1 \times Z^*$ will be called an optimal solution of (7.2) if it is feasible and if

ψ is equal to the infimum in (7.2) Z^* being the dual space of the normed space Z.

Since $(R^1 \times Z^*)$ is the dual space of $(R^1 \times Z)$, for every $(\eta^*, u^*) \in R^1 \times Z^*$, and every $\psi \in R^1$, the expression:



$$\eta^*\phi + \langle u^*, z\rangle + \psi = 0,$$

can be interpreted as a hyperplane in $R^1 \times Z$. Taking in account that in the restrictions of (7.2) $\eta := -1$, problem (7.2) may be considered as a process of seeking a non-vertical supporting hyperplane of the set Γ, which intersects the vertical axis $z := 0$ at a point as low as possible. This geometric interpretation suggests that the supremum in (3.2) cannot be higher than the infimum in (7.2) and that, generally, the supremum in (3.2) will be equal to the infimum in (7.2). The meaning of these relations is obvious: if $(\phi, 0)$ and (ψ, u^*) satisfy the restrictions of (3.2) and (7.3) then $\phi = \psi$ will imply that $(\phi, 0)$ and (ψ, u^*) are both optimal solutions; so, in that case, the equality $\phi = \psi$ gives a necessary condition for optimality.

<u>8. Proposition.</u>

If the problems (3.2) and (7.2) both possess a feasible solution, then the infimum in (7.3) is not smaller than the supremum in (3.7). Moreover, in that case they are both finite.

<u>Proof.</u> This property follows immediately from the restrictions of the infimum problem (7.2).

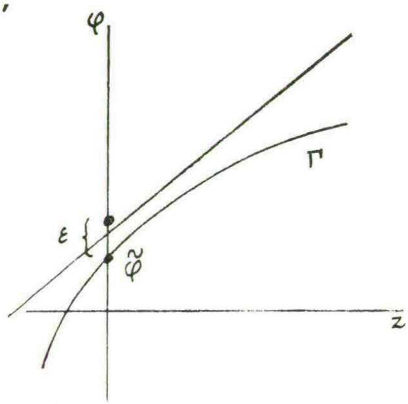
9. Proposition.

Suppose $\Gamma \cap (R^1 \times \{0\})$ is non-empty, and suppose that the supremum in (3.2) is bounded. Then, the closedness of set $\Gamma_+ := \Gamma \cap (R^1 \times Z_+)$ implies:

a) Infimum problem (7.2) possesses a feasible solution.

b) The infimum in (7.2) is equal to the supremum in (3.2).

Proof: By virtue of proposition 5, the suppositions imply that for all positive numbers $(\tilde{\phi}+\varepsilon,0) \notin \Gamma$, $\tilde{\phi}$ being the supremum in (3.2). Since $\bar{\Gamma}$ is closed and convex (viz. 4-a), this implies (see for instance ref. 3, page 134) for every $\varepsilon > 0$ the existence of a closed half-space $\tilde{\Gamma}_\varepsilon$ such that $\bar{\Gamma} \subset \tilde{\Gamma}_\varepsilon$, $(\tilde{\phi}+\varepsilon) \notin \tilde{\Gamma}_\varepsilon$. Every closed half-space $\tilde{\Gamma}_\varepsilon$ can be expressed by:



$$\tilde{\Gamma}_\varepsilon := \{(\phi,z) \in R^1 \times Z \,|\, \eta^*_\varepsilon \phi + \langle u^*,z\rangle + \psi_\varepsilon \geqq 0\}, \tag{9.1}$$

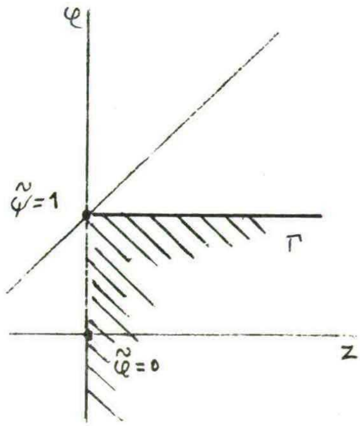
where $(\eta^*_\varepsilon, u^*_\varepsilon) \in R^1 \times Z^*$ and $\psi_\varepsilon \in R^1$ further to be determined. First of all, we observe that $\eta^*_\varepsilon < 0$; for if $\eta^*_\varepsilon \geqq 0$, then (9.1) and $\bar{\Gamma} \subset \tilde{\Gamma}_\varepsilon$ would imply: $(\tilde{\phi}+\varepsilon,0) \in \tilde{\Gamma}_\varepsilon$. So, since the expression $\eta^*_\varepsilon \phi + \langle u^*_\varepsilon,z\rangle + \psi_\varepsilon$ in (9.1) is homogeneous in η^*_ε, u^*, and ψ_ε, we may choose these quantities in such a manner that $\eta^*_\varepsilon = -1$. Thus we may conclude that, for every $\varepsilon > 0$, a point $(\psi_\varepsilon, u^*) \in R^1 \times Z^*$ exists such that:

$$\left.\begin{aligned} \langle u^*,z\rangle - \phi + \psi_\varepsilon \geqq 0, &\text{ for all } (\phi,z) \in \bar{\Gamma} \\ \psi_\varepsilon \leqq \tilde{\phi} + \varepsilon & \end{aligned}\right\} \tag{9.2}$$

The first inequality in (9.2) implies the a-part of this proposition. The b-part follows from the validity of (9.2) for all $\varepsilon > 0$, and from inequality $\tilde{\phi} \leqq \psi_\varepsilon$ (viz. proposition 8).

10. Example.

In the example of §6, one may deduce that $(\psi, u_1^*, u_2^*) := (1,1,1)$ is a feasible solution of the dual problem (7.2). Since $[-\infty,1] \times \{0\} = \bar{\Gamma} \cap R^1 \times Z_+$, the infimum $\tilde{\psi}$ of (7.2) is equal to one. This implies that $(\psi, u_1^*, u_2^*) := (1,1,1)$ is an optimal solution of the infimum problem (7.2). So, in this case the supremum in (3.2) is definitely smaller than the infimum in (7.2). This phenomenon is known as the "duality gap".



We observe that proposition 9 does not include any statement with respect to the existence of an optimal solution of the infimum problem (7.2). However, such a statement can be given by strengthening the conditions.

11. Proposition.

If $\mathrm{int}(\Gamma) \cap (R^1 \times \{0\})$ is non-empty and if the supremum in (3.2) is bounded, then:

a) The infimum problem (7.2) possesses an optimal solution.

b) The infimum in (7.2) is equal to the supremum in (3.2).

Proof: The definition of Γ (viz. §3) implies $(\tilde{\phi},0) \in \bar{\Gamma}$ $(\tilde{\phi},0) \notin \mathrm{int}(\Gamma)$, $\tilde{\phi}$ being the supremum in (3.2). Since Γ is convex and since $\mathrm{int}(\Gamma) \neq \emptyset$, we may conclude from $(\tilde{\phi},0) \notin \mathrm{int}(\Gamma)$ that a closed half-space $\tilde{\Gamma}$ exists such that

$\Gamma \subset \tilde{\Gamma}$, $(\tilde{\phi},0) \notin \mathrm{int}(\tilde{\Gamma})$ and $(\tilde{\phi},0) \in \tilde{\Gamma}$ (viz. ref. 3, page 133).
Expressing this half-space in the form
$\tilde{\Gamma} := \{(\phi,z) \in R^1 \times Z \mid \eta^*\phi + \langle u^*,z\rangle + \psi \geqq 0\}$, where
$(\eta^*,u^*) \in R^1 \times Z^*$ and $\psi \in R^1$, we may conclude that
$(\tilde{\phi},0) \notin \mathrm{int}(\tilde{\Gamma})$, together with $(\tilde{\phi},0) \in \tilde{\Gamma}$ implies: $\eta^*\tilde{\phi} + \psi = 0$.
Since $[-\infty,\tilde{\phi}] \times \{0\} \subset \tilde{\Gamma}$, the latter implies $\eta^* \leqq 1$. Assumption $\mathrm{int}(\Gamma) \cap (R^1 \times \{0\}) \neq \emptyset$ and $\Gamma \subset \tilde{\Gamma}$ imply: $\mathrm{int}(\tilde{\Gamma}) \cap (R^1 \times \{0\}) \neq \emptyset$.
So, a $\bar{z} \in Z$ and a $\bar{\phi} < \tilde{\phi}$ exists such that $(\bar{\phi},\bar{z}) \in \tilde{\Gamma}$, and
$\langle u^*,z\rangle < 0$. Since $\eta^* \leqq 1$, the latter implies, by virtue of the definition of $\tilde{\Gamma}$: $\eta^* < 0$. Putting $(\bar{\phi},\bar{u}) := -(\frac{1}{\eta^*})\ (\psi,u^*)$,
we may conclude:

$$\left.\begin{aligned} &\langle\bar{u},z\rangle - \phi + \bar{\psi} \geqq 0, \text{ for all } (\phi,z) \in \Gamma \\ &\bar{\psi} = \tilde{\phi} \end{aligned}\right\} \qquad (11.1)$$

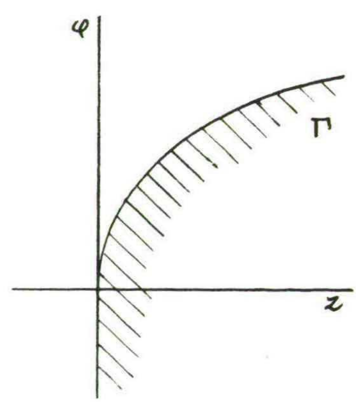
Since the infimum in (7.2) cannot be smaller than the supremum in (3.2) (viz. proposition 8), (11.1) proves both the a-part of this proposition, and the b-part.

12. Example.

The significance of the condition that $\mathrm{int}(\Gamma) \cap (R^1 \times Z)$ is not empty can be given by the problem

$$\phi(z) := \sup_x \sqrt{x} \left| \begin{array}{l} x \leqq 0 + z \\ x \geqq 0 \end{array}\right.$$

Since for all $z \geqq 0$: $\phi(z) = \sqrt{z}$, the vertical axis $z = 0$ is tangent to set Γ. The corresponding dual problem has no optimal solution, but the infimum is equal to zero.



A more interesting, but rather complicated, example can be given by the problem:

$$\sup \sum_{t=1}^{\infty} (0.9)^t x_2(t) \left| \begin{array}{ll} \frac{1}{2}x_1(t+1)+x_2(t+1)-x_1(t) & \leqq -0.1 \\ x_1(t+1) & \leqq 1 \\ x(t+1) \geqq 0 & \\ x_1(0) = 0.2 & \end{array} \right\} t = 0,1,\ldots$$

This problem demonstrates similar phenomena as the preceding example (viz. ref. 2). Next, we shall deduce some duality relations in terms of the original problem (3.1).

13. Proposition.

A point $(\psi,u^*) \in R^1 \times Z^*$ is a feasible solution of the infimum problem (7.2) if, and only if:

$$\left. \begin{array}{l} \langle u^*,G[x]\rangle - q[x] \geqq \langle u^*,g\rangle - \psi, \text{ for all } x \in X_+ \\ \langle u^*,g\rangle \leqq \psi \\ u^* \in Z^*_+ \end{array} \right\}, \qquad (13.1)$$

Z^*_+ being the positive cone of the dual space of Z, defined by: $Z^*_+ := \{z^* \in Z^* \mid \langle z^*,z\rangle \geqq 0, \text{ for all } z \in Z_+\}$.

Proof: The definitions of Γ (viz. §3) and of problem (7.2) imply successively the equivalence of the statements:

$(\psi,u^*) \in R^1 \times Z^*$ is a feasible solution of (7.2)

$\langle u^*,z\rangle - \phi+\psi \geqq 0$, for all $(\phi,z) \in \Gamma$,

$\langle u^*,G[x]+y-g\rangle - q[x]+\psi \geqq 0$, for all $x \in X_+$, $y \in Z_+$,

$\langle u^*,G[x]\rangle - \langle u^*,g\rangle + \langle u^*,y\rangle - q[x]+\psi \geqq 0$, for all $x \in X_+$, $y \in Z_+$

(13.2)

- Putting x:= 0, y:= 0, (13.2) implies by virtue of assumption 3-e:

  $\langle u^*, g\rangle \leqq \psi$.

- Putting x:= 0, (13.2) implies

  $\langle u^*, y\rangle \geqq \langle u^*, g\rangle - \psi$, for all $y \in Z_+$.

  This is possible only if $u^* \in Z_+^*$.

- Putting y:= 0, (13.2) implies

  $\langle u^*, G[x]\rangle - q[x] \geqq \langle u^*, g\rangle - \psi$, for all $x \in X_+$.

The last three statements affirm the necessity of (13.1). The sufficiency of (13.1) immediately follows from the fact that (13.1) implies (13.2), which is a sufficient condition for $(\psi, u^*) \in R^1 \times Z^*$ to be a feasible solution of (7.2).

14. The (direct) dual problem.

The latter proposition gives rise to the following programming problem in the dual space of Z:

$$\hat{\psi} := \inf_{\mu, u^*} \ \mu + \langle g, u^*\rangle \left| \begin{array}{l} \langle u^*, G[x]\rangle + \mu \geqq q[x], \text{ for all } x \in X_+ \\ u^* \in Z_+^*, \ \mu \in R_+^1 \end{array} \right. \tag{14.1}$$

(μ, u^*) is called a <u>feasible solution of (14.1)</u> if (μ, u^*) satisfies the conditions of (14.1). $(\hat{\mu}, \hat{u}^*)$ is called an <u>optimal solution of (14.1)</u> if it is a feasible solution for which $\hat{\mu} + \langle \hat{u}^*, g\rangle = \hat{\psi}$.

Clearly, the definition of infimum problem (7.2) and proposition (13) imply:

a) (μ, u^*) is a feasible solution of (14.1) if, and only if, (ψ, u^*), with $\psi := \mu + \langle u^*, g\rangle$, is a feasible solution of (7.2).

b) The infimum in (14.1) is equal to the infimum in (7.2).

c) (μ,u^*) is an optimal solution of (14.1) if, and only if, (ψ,u^*), with $\psi := \mu+\langle u^*,g\rangle$, is an optimal solution of (7.2).

15. Theorem.

Consider the programming problem

$$\hat{\phi} := \sup_x q[x] \left| \begin{array}{l} G[x] \leqq g \\ x \in X_+, \end{array} \right. \tag{15.1}$$

defined in §3, and its corresponding dual problem

$$\hat{\psi} := \inf_{\mu,u^*} \mu+\langle u^*,g\rangle \left| \begin{array}{l} \langle u^*,G[x]\rangle + \mu \geqq q[x], \text{ for all } x \in X_+ \\ u^* \in Z^+_+, \ \mu \in R^1_+, \end{array} \right. \tag{15.2}$$

defined in §14. Suppose these problems both possess feasible solutions. Then:

a) For every feasible solution $x \in X$ of (15.1) and every feasible solution $(\mu,u^*) \in R^1 \times Z^*$ of (15.2):

$$q[x] = \mu+\langle u^*,g\rangle - \langle u^*,y\rangle - v^*[x], \tag{15.3}$$

where $y := g-G[x]$, and where $v^*[x] := \langle u^*,G[x]\rangle - q[x]+\mu$.

b) For every feasible solution $x \in X$, $(\mu,u^*) \in R^1 \times Z^*$ of (15.1) and (15.2):

$$q[x] \leqq \mu+\langle u^*,g\rangle$$

c) The infimum in (15.2) and the supremum in (15.1) are bounded.

d) If the supremum in (15.1) is equal to infimum in (15.2), then feasible solutions $x \in X$ and (μ,u^*) of (15.1) and (15.2) resp. both are optimal if and only if:

$$\langle u^*,y\rangle = 0, \ v^*[x] = 0, \tag{15.4}$$

where $y := g-G[x]$, and where $v^*[x] := \langle u^*,G[x]\rangle - q[x]+\mu$.

Proof:

In order to prove the a-part, let $x \in X$ and let (μ,u^*) be feasible solutions of (15.1) and (15.2). Then, $y := g-G[x]$ and $v^*[x] := <u^*,G[x]>-q[x]+\mu$ implies: $q[x] = <u^*,G[x]>-v^*[x]+\mu =$ $= <u^*,g-y>-v^*[x]+\mu = <u^*,g>-<u^*,y>-v^*[x]+$: which proves the a-part. The b- and c-part, immedeately follow from a.

Property 15-b implies for every feasible solution $x \in X$ and (μ,u^*): $q[x] \leqq +\mu<u^*,g>$. If these feasible solution satisfy (15.4), then (15.3) implies $q[x] = \mu+<u^*,g>$. Clearly, both are optimal solutions. If the supremum in (15.1) is equal to the infimum in (15.2), then feasible solutions x, (μ,u^*) are both optimal, only if: $q[x] = <u^*,g>$. Since $<u^*,y> \geqq 0$ and $v^*[x] \geqq 0$, property 15-a implies that (15.4) is a necessary condition for optimality.

16. Duality relation in linear programming problems.

In case $G[\cdot]: X_+ \to Z$ and $q[\cdot]: X_+ \to R^1$ are bounded linear functions on X, the relations of theorem 15 take the form of the well known duality relations appearing in linear programming. We denote the adjoint operator of G by G^*, which is defined as a function $G^*: Z^* \to X^*$, with the property that for every $x \in X$, $u^* \in Z^*$: $<u^*,G[x]> = <x,G^*[u^*]>$. Defining $X_+^* := \{x^* \in X^* \mid <x^*,x> \geqq 0$, for all $x \in X_+\}$, one may verify that linearity implies: $(\mu,u^*) \in R_+^1 \times Z_+^*$ satisfies $(<u^*,G[x]>+\mu \geqq$ $\geqq q[x]$, for all $x \in X_+)$ if, and only if, $G^*[u^*] \geqq q$. So, problem (15.1), (15.2) can be written:

$$\hat{\phi} := \sup_{(x,y)} <q,x> \left| \begin{array}{l} G[x] + y = g \\ x \in X_+,\ y \in Z_+ \end{array} \right. \tag{16.1}$$

$$\hat{\psi} := \inf_{(u,v)} <g,u^*> \left| \begin{array}{l} G^*[u^*] - v^* = q \\ u^* \in Z_+^*,\ v^* \in X_+^*. \end{array} \right. \tag{16.2}$$

If these problems both possess feasible solutions, then:

a) The infimum and the supremum are bounded.

b) $\hat{\phi} \leqq \hat{\psi}$.

c) For every feasible $(x,y) \in X \times Z$ and every feasible $(u^*,v^*) \in Z^* \in X^*$: $<q,x> = <g,u^*>-<u^*,y>-<v^*,x>$. (16.3)

d) If $\hat{\phi} = \hat{\psi}$, then feasible solutions $(x,y) \in X \times Z$ and $(u^*,v^*) \in Z^* \in X^*$ both are optimal if and only if: $<u^*,y> = 0$, $<v^*,x> = 0$. (16.4)

17. Supporting linear programming problems.

The simplicity of the duality relations in linear programming also can be obtained in convex programming by constructing a linear approximation in an optimal point. To that end we consider the following linear programming problem:

$$\bar{\phi} := \sup_{(x,y)} <\hat{q},x> \;\Big|\; \hat{G}[x]+y = \hat{g},\ x \in X_+,\ y \in Z_+ \qquad (17.1)$$

where $\hat{G}[\cdot] : X \to Z$, $\hat{q}[\cdot] : X \to R^1$ are bounded linear functionals. The normed vector spaces X and Z are the same as the original convex programming problem (3.1) (or 15.1). This problem is called a supporting linear programming problem in an optimum point $\hat{x} \in X_+$ of convex programming problem (3.1) if sumultaneously:

a) $\forall\, x \in X_+$: $\hat{G}[x]-\hat{g} \leqq G[x]-g$ (which implies that every feasible solution of (3.1) is a feasible solution of (17.1)).

b) $\hat{G}[\hat{x}]-\hat{g} = G[\hat{x}]-g$

c) $\forall\, x \in X_+$: $<\hat{q},x> - <\hat{q},\hat{x}> \geqq q[x]-q[\hat{x}]$

d) $\hat{x}$ is an optimal solution of (17.1)

Applying the statements of §16, the dual problem of (17.1) becomes:

$$\bar{\psi} := \inf_{(u,v)} <\hat{g},u^*> \;\Big|\; \hat{G}^*[u^*]-v^* = \hat{q},\ u^* \in Z^*_+,\ v \in X^*_+. \qquad (17.2)$$

Now, we can formulate the following properties:

e) If (u^*,v^*) is a feasible solution of dual problem (17.2), then, for every $\mu \geqq \max\{0,(<u^*,\hat{g}-g>-<\hat{q},\hat{x}>+q[\hat{x}])\}$: (μ,u^*) is a feasible solution of the original dual problem (15.2). (Proof follows from def. (17.2) and from the conditions (a) and (c)).

f) Suppose problem (17.2) possesses a feasible solution, and suppose the inf. in (17.2) is equal to the sup. in (17.1). Then, for every optimal solution $(\tilde{u},\tilde{v})$ of (17.2), the point $(\tilde{\mu},\tilde{u})$ where $\tilde{\mu}:= q[\hat{x}]-\langle\tilde{u},g\rangle$, is an optimal solution of the original dual problem (15.2).

Proof: Let $(\tilde{u},\tilde{v})$ and $(\hat{x},\hat{y})$ be optimal solutions of (17.2) and (17.1) resp. Then, the definitions of (17.1), (17.2), property 16-d, and the conditions 17-a,c imply, for every $x \in X_+$:

$\langle\tilde{u},G[x]\rangle-q[x] \geqq \langle\tilde{u},\hat{G}[x]\rangle-\langle\hat{q},x\rangle + \langle\tilde{u},G[\hat{x}]\rangle-q[\hat{x}]$.

This implies for every $\lambda \in R^1_+$:

$\langle\tilde{u},G[\lambda x]\rangle-q[\lambda x] \geqq \langle\tilde{u},G[\hat{x}]\rangle-q[\hat{x}]$ (by putting $x:= \lambda\hat{x}$).

Convexity of $(G[\cdot]-q[\cdot])$ and $G[0] = 0$, $q[0] = 0$ implies:

$\forall\ \lambda \in [0,1]: \langle\tilde{u},G[\lambda\hat{x}]\rangle-q[\lambda\hat{x}] \leqq \lambda\{\langle\tilde{u},G[\hat{x}]\rangle-q[\hat{x}]\}$.

The latter inequalities imply: $\langle\tilde{u},G[\hat{x}]\rangle \leqq q[\hat{x}]$, so that, by supposition (b) and by $\langle\tilde{u},\hat{G}[\hat{x}]-\hat{g}\rangle = 0$:

$\langle\tilde{u},g\rangle \leqq q[\hat{x}]$. Since $\langle u^*,\hat{g}\rangle = \langle\hat{q},\hat{x}\rangle$ (by the optimality of u^* and $\hat{x}$), the latter implies (viz. property e) that $(\tilde{\mu},\tilde{u})$, with $\tilde{\mu}:= q[\hat{x}]-\langle\tilde{u},g\rangle$, is a feasible solution of (15.2).

Since, in addition $(\tilde{\mu},\tilde{u})$ satisfies the conditions of 15-d, it is an optimal solution of (15.2), as well.

Stronger relations between the original problem (3.1) and a corresponding supporting linear programming problem (17.1) can be obtained when we assume that the functions are differentiable in some way. We shall call a functions $G[\cdot]: X_+ \to Z$ <u>differentiable in a point $\hat{x} \in X_+$ with respect to its domain X_+</u>, if a bounded linear function $\nabla G[\cdot]: X \to Z$ exists such that for every $x \in X_+$: $\frac{1}{\alpha}\{G[\hat{x}+\alpha x]-G[\hat{x}]\}= \nabla G[x]$, $\alpha \to 0$. (17.3)

The bounded linear function $\nabla G[\cdot]$ is called the <u>derivative</u> of $G[\cdot]$ in point $\hat{x}$ with respect to X_+. (This concept is closely related to the Fréchet derivative of a function viz. ref. 3 page 175). A similar notion can be given for the functional $q[\cdot]: X_+ \to R^1$. Now, we can formulate the following theorem:

18. Theorem.

If problem (viz. def. §3):

$$\sup q[x] \,|\, G[x] \leqq g,\ x \in X_+, \tag{18.1}$$

satisfies: (a) $\text{int}(Z_+) \neq \emptyset$ and a feasible solution $\bar{x} \in X$ exists such that $g-G[\bar{x}] \in \text{int}(Z_+)$, (b) an optimal point $\hat{x}$ exists for which $G[\cdot]$, $q[\cdot]$ are differentiable with respect to X_+, then the linear programming problem:

$$\bar{\phi} := \sup_x \langle \nabla q, x\rangle \,\Big|\, \nabla G[x] \leqq \hat{g} := g-G[\hat{x}]+\nabla G[\hat{x}],\ x \in X_+ \tag{18.2}$$

$\nabla G[\cdot]$, $\nabla q[\cdot]$ being derivatives of $G[\cdot]$ and $q[\cdot]$ in the optimal point $\hat{x}$ of condition (b), is a supporting linear programming problem.

Proof: The convexity of $G[\cdot]$ implies: for every $\bar{x}$, $x \in X_+$, $\alpha \in]0,1[$: $G[\bar{x}+\alpha x] \leqq \alpha G[\bar{x}+x] + (1-\alpha)G[\bar{x}]$ and successively: $(\frac{1}{\alpha})\{G[\bar{x}+\alpha x] - G[\bar{x}]\} \leqq G[\bar{x}+x] - G[\bar{x}]$. Let $\nabla G_{\bar{x}}[\cdot]$ be the derivative in point $\bar{x} \in X_+$ (def. §17), then (17.8) and the latter inequality imply:

$$G[\bar{x}+x] \geqq G[\bar{x}] + \nabla G_{\bar{x}}[x], \text{ for every } x \in X_+. \tag{18.3}$$

In a similar manner, the concavity of $q[\cdot]$ implies:

$$q[\bar{x}+x] \leqq q[\bar{x}] + \nabla q_{\bar{x}}[x], \text{ for every } x \in X_+, \tag{18.4}$$

$\nabla q_{\bar{x}}[\cdot]$ being the derivative in a point $\bar{x} \in X_+$ (def. §17).
Putting $\bar{x} := \hat{x}$ ($\hat{x}$ being an optimal solution as supposed in (a) and (b)), a straightforward calculation will show that (18.3) and (18.4) imply that problem (18.2) satisfies the conditions 17-a, b, c.

In order to prove that $\hat{x}$ is an optimal solution of (18.2), assume that $\underline{x}$ is a feasible solution of (18.2) such that $\langle \nabla q, \underline{x}\rangle - \langle \nabla q, \hat{x}\rangle := \delta > 0$.
Defining for every number $\alpha \in]0,1[$ the functions:

$$\left.\begin{aligned} \beta(\alpha) &:= (\tfrac{1}{\alpha})\{G[\hat{x}+\alpha(\underline{x}-\hat{x})] - G[\hat{x}]\} \\ \gamma(\alpha) &:= (\tfrac{1}{\alpha})\{q[\hat{x}+\alpha(\underline{x}-\hat{x})] - q[\hat{x}]\} \end{aligned}\right\} , \tag{18.5}$$

the fact that G[$\cdot$] and q[$\cdot$] are differentiable in $\hat{x}$ (def. §17) implies:

$$\lim_{\alpha\to 0} \beta(\alpha) = 0, \quad \lim_{\alpha\to 0} \gamma(\alpha) = 0. \tag{18.6}$$

Starting from the definitions (18.2) and (18.5), a straightforward calculation will show that for every $\alpha\in]0,1[$:

$$\left.\begin{aligned} G[(1-\alpha)\hat{x}+\alpha\underline{x}] &\leqq g + \alpha\beta(\alpha) \\ q[(1-\alpha)\hat{x}+\alpha\underline{x}] &= q[\hat{x}] + \alpha\delta + \alpha\gamma(\tfrac{1}{\alpha}) \end{aligned}\right\} , \tag{18.7}$$

where $\delta := \langle\nabla q,\underline{x}\rangle - \langle\nabla q,\hat{x}\rangle$ being, by assumption, positive.

Let $\bar{x} \in X_+$ be a feasible solution of (18.1) which satisfies condition (a) of this theorem; i.e. $g-G[\bar{x}]\in \mathrm{int}(Z_+)$. Then, for every $\alpha\in]0,1[$ a number $\varepsilon(\alpha)$ can be defined by:

$$\varepsilon(\alpha) := \inf_{\underset{\sim}{\varepsilon}} \underset{\sim}{\varepsilon} \;\Big|\; \underset{\sim}{\varepsilon}(g-G[\bar{x}]) \geqq \beta(\alpha),\; \underset{\sim}{\varepsilon}\in[0,\infty]. \tag{18.8}$$

Combining (18.6) and (18.8), we find $\lim_{\alpha\to 0} \varepsilon(\alpha) = 0$, which implies the existence of an interval $]0,\bar{\alpha}[\; \subset\;]0,1[$ such that, for every $\alpha\in]0,\bar{\alpha}]$: $\alpha\varepsilon(\alpha)\in[0,1]$. Now, defining for every $\alpha\in]0,\bar{\alpha}]$ a vector $\tilde{x}(\alpha) := [1-\alpha\varepsilon(\alpha)][(1-\alpha)\hat{x} + \alpha\underline{x}] + \alpha\varepsilon(\alpha)\bar{x}$, a straightforward calculation shows that (18.7) and (18.8) implies:

$$\left.\begin{aligned} G[\tilde{x}(\alpha)] &\leqq g \\ q[\tilde{x}(\alpha)] &\geqq \{1-\alpha\varepsilon(\alpha)\}\{q[\hat{x}] + \alpha\delta + \alpha\gamma(\alpha)\} + \alpha\varepsilon(\alpha)q[\bar{x}] \end{aligned}\right\}$$

Since, by assumption, $\delta > 0$, the latter implies, by virtue of (18.5), the existence of an $\underline{\alpha}\in]0,\bar{\alpha}[$ such that $q[\tilde{x}(\underline{\alpha})] > q[\hat{x}]$. Since $\tilde{x}(\underline{\alpha})$ is a feasible solution of (18.1), this conflicts the supposition that $\hat{x}$ is an optimal solution of (18.1). Thus we may conclude that the supporting linear programming problem

(18.2) does <u>not</u> possess a feasible solution $\underline{x}$ such that $\langle \nabla q, \underline{x} \rangle > \langle \nabla q, \hat{x} \rangle$. Thus, we may conclude: $\hat{x}$ is an optimal solution of problem (18.2).

<u>19. Example.</u>

The meaning of condition 18-a, may be illustrated by the examples:

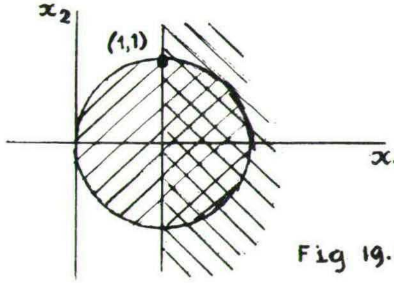


Fig 19.1

$$\text{a)} \quad \sup_{x,y} y \;\middle|\; \begin{array}{l} (x-1)^2 + y^2 \leqq 1 \\ x \geqq 1 \\ y \geqq 0 \end{array}$$

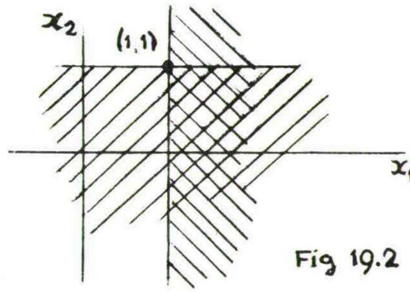


Fig 19.2

$$\text{b)} \quad \sup_{x,y} y \;\middle|\; \begin{array}{l} (x-1)^2 + y^2 \leqq 1 \\ x \geqq 2 \\ y \geqq 0 \end{array}$$

Clearly $(\hat{x},\hat{y}) := (1,1)$ is an optimal solution of problem (a) (viz. fig. 19.1). The supporting linear problem takes the form: $\{\sup y \mid y \in [0,1],\ x \geqq 1\}$ (viz. fig. 19.2).

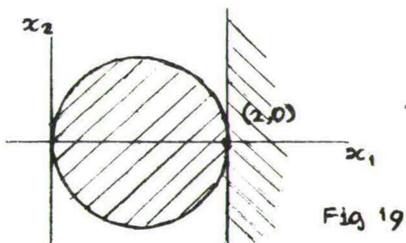


Fig 19.3

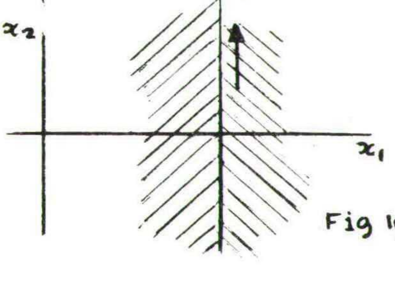


Fig 19.4

The only feasible solution of (b) is: $(\hat{x},\hat{y}) := (2,0)$, so it is optimal as well. For this problem there is no supporting linear problem. For linearization in $(\hat{x},\hat{y}) := (2,0)$ would result in the problem: $\{\sup y \mid x \geqq 2,\ x \leqq 2,\ y \geqq 0\}$. Clearly, for this problem, the supremum is unbounded. We observe that (a) satisfies condition 18-a, but (b) does not.

20. Remark.

The consequences of linearization will be discussed later (§23). First of all we wish to introduce some direct conditions which imply the existence of optimal solutions and the equality of the supremum of the original problem and the infimum of the dual problem. The propositions 9 and 11, both give suitable starting points. For application of proposition 9 we have to prove that set $\Gamma_+ := \Gamma \cap (R^1 \times Z_+)$ is closed. We shall do so in two different ways. In the first method we assume that the closed unit sphere in X is compact. Since some vector spaces (for instance l_1 and l_∞) do not possess this property, we assume in the second way that the closed unit sphere in X is weak* compact. (weak* convergency viz. ref. 3 , page 127).

21. Theorem.

Consider the programming problem $\{\sup q[x] \mid G[x] \leqq g,\ x \in X_+\}$ defined in §3, which possesses a feasible solution. Suppose numbers M_1, M_2 exist such that for every $x \in X_+$, $z \in Z_+$ with $G[x] \leqq g+z$, there is an $\bar{x} \in X_+$ satisfying $\|x\| \leqq M_1+M_1 \|z\|$, $G[\bar{x}] \leqq g+z$, $g[\bar{x}] \geqq q[x]$. Suppose the closed unit sphere in X is compact, or, suppose the closed unit sphere in X is weak* compact, the positive cones X_+, Z_+ are weak* closed, and the functions $G[\cdot]$, $q[\cdot]$ are weak* continuous.

Then: the problem possesses an optimal solution, its dual problem (14.1) possesses a feasible solution, and the supremum in (3.1) is equal to the infimum in (14.1).

Proof: Let $\Gamma \subset R^1 \times Z$ be the set defined in §3 and let $\{(\phi_i, z_i)\}_1^\infty \in \Gamma_+ := \Gamma \cap R^1 \times Z_+$ be a sequence which converges to a point $(\phi_0, z_0) \in R^1 \times Z$ (the existence of such a sequence is implied by the presumption that 3.1 possesses a feasible solution). Let M_1, M_2 be numbers as mentioned in the suppositions of this theorem. Then, $\{(\phi_i, z_i)\}_1^\infty \subset \Gamma_+$ implies the existence of a sequence $\{x_i\}_1^\infty \subset X_+$ satisfying:

$$\left.\begin{aligned} G[x_i] &\leqq g+z_i \\ \|x_i\| &\leqq M_3 := M_1 + M_2 \{\sup_k \|z_k\|\} \\ q[x_i] &\geqq \phi_i \end{aligned}\right\} \quad t = 1,2,\ldots \tag{21.1}$$

Now, consider the case that the closed unit sphere in X is compact. Then, $\|x_i\| \leqq M_3$ $i = 1,2,\ldots$ implies the existence of a subsequence $\{x_{i(k)}\}_{k=1}^{\infty}$ which converges to a point $x_0 \in X$ with $\|x_0\| \leqq M_3$. Since the functions $G[\cdot]$ and $q[\cdot]$ are supposed to be continuous (viz. 3-c) and since the positive cones X_+ and Z_+ are supposed to be closed, the latter implies: $G[x_0] \leqq g+z$, $q[x_0] \geqq \phi_0$, $z_0 \in Z_+$, $x_0 \in X_+$. Clearly, $(\phi_0, z_0) \in \Gamma_+$. Thus, we may conclude that $\{(\phi_i, z_i)\}_1^{\infty} \subset \Gamma_+$, $\{(\phi_i, z_i) \to (\phi_0, z_0), i \to \infty\}$ implies $(\phi_0, z_0) \in \Gamma_+$, which proves the closedness of Γ_+. Moreover, $\phi_0 \leqq q[x_0]$, $\|x_0\| \leqq M_3$ and the boundedness of $q[\cdot]$ (viz. condition 3-b) imply the boundedness of the supremum in (3.1). So, by virtue of 5-b, 9, and of 14-a,b, we may conclude: problem (3.1) possesses an optimal solution, the dual problem (14.1) possesses a feasible solution, and, finally, the infimum in (14.1) is equal to the supremum in (3.1).

Using the concepts of weak* convergency instead of the concepts related to convergency, the remaining part of the theorem may be proved in a similar manner.

22. Theorem.

Consider the programming problem $\{\sup q[x] \mid G[x] \leqq g,\ x \in X_+\}$ defined in §3. Suppose: (1) $\mathrm{int}(Z_+) \neq \emptyset$, (2) there is an $\bar{x} \in X_+$ such that $G[\bar{x}] < g$, (3) the supremum in (3.1) is bounded. Then;

a) Dual problem $\{\inf \mu + \langle u^*, g\rangle \mid (\mu, u^*) \in R^1_+ \times Z^*_+$, $\langle u^*, G[x]\rangle + \mu \geqq q[x]$, for all $x \in X_+\}$ possesses an optimal solution.

b) The infimum of the dual problem is equal to the supremum in (3.1).

<u>Proof:</u> Let $\Gamma \in R^1 \times Z$ be the set defined in §3. Then, the suppositions (1) and (2) imply: $\mathrm{int}(\Gamma) \cap (R^1 \times \{0\}) \neq \emptyset$.
By virtue of proposition 11 and of 14-a, b, c, the latter and the boundedness of the supremum (supposition 3) imply both the a- and b- part of this theorem.

<u>23. Corollary of 18 and 22 (Generalized Kuhn-Tucker Theorem).</u>

Let $\{\hat{\phi} := \sup q[x] \mid G[x] \leqq g,\ x \in X_+\}$ be the programming problem defined in §3. Suppose (1) $\mathrm{int}(Z_+) \neq \emptyset$, (2) and $\bar{x} \in X_+$ exists such that $G[\bar{x}] < g$. Let $\hat{x} \in X$ be an optimal solution of this problem, for which the functions $G[\cdot]$ and $q[\cdot]$ are differentiable with respect to X_+ (def. §17). Then vectors $\hat{u}^* \in Z^*_+$, $\hat{v}^* \in X^*_+$ exist such that the Lagrange function:

$$L[x] := q[x] + \langle \hat{v}^*, x \rangle + \langle \hat{u}^*, g - G[x] \rangle, \tag{23.1}$$

is stationary at $\hat{x}$; and, in addition, such that

$$\langle \hat{u}^*, g - G[\hat{x}] \rangle = 0, \quad \langle \hat{v}^*, \hat{x} \rangle = 0 \,. \tag{23.2}$$

<u>Proof:</u> Let $\{\sup \langle \nabla q, x \rangle \mid \nabla G[x] \leqq \hat{g} := g - G[\hat{x}] + \nabla G[\hat{x}],\ x \geqq 0\}$ be a supporting linear programming problem in point $\hat{x}$ (def. §17). Let $\bar{x} \in X_+$ be the vector of supposition (2); then (by 18 and 17-a) $\nabla G[\bar{x}] < \hat{g} := g - G[\hat{x}] + \nabla G[\hat{x}]$. Applying theorem 22 on the supporting linear programming problem, the latter implies the existence of an optimal solution $(\hat{u}^*, \hat{v}^*) \in Z^* \times X^*$ for $\{\inf \langle \hat{g}, u^* \rangle \mid \nabla G^*[u^*] - v^* = \nabla q,\ u^* \in Z^*_+,\ v^* \in X^*_+\}$, which is (viz. §16) the dual problem of the supporting linear problem. Since $\hat{x}$ is an optimal solution of the supporting problem, as well, (viz. 17-d), we have (viz. 16-d): $\langle \hat{u}^*, g - G[\hat{x}] \rangle = 0$, $\langle \hat{v}^*, \hat{x} \rangle = 0$. Further, $\nabla G^*[\hat{u}^*] - \hat{v}^* = \nabla q$ implies the stationarity of the Lagrange function defined by (23.1).

24. Conditions for the existence of dual optimal solutions.

For some spaces, it is very hard to characterize the dual space (for instance dual space of l_∞). Therefore it can be useful to study the dual problem in a slightly modified form;

$$\inf_{(\mu,u)} \mu+\langle h,u\rangle \left| \begin{array}{l} \langle H[w],u\rangle+\mu \geqq r[w], \text{ for all } w \in W_+ \\ (\mu,u) \in R^1_+ \times U_+ \end{array} \right. \tag{24.1}$$

The quantities are specified as follows:

- U and W; normed vector spaces; U_+, W_+ closed positive cones.
- $H[\cdot] : W_+ \to U^*$; continuous and bounded.
- $r[\cdot] : W_+ \to R^1$; continuous and bounded.

A point $(\mu,u) \in R^1 \times U$ is called a feasible solution of (24.1) if it satisfied the restrictions of (24.1); this point is called an optimal solution if it is feasible and if, in addition, $\mu+\langle h,u\rangle$ is equal to the infimum in (24.1).

If such a programming problem possesses a feasible solution then the following proposition can be given: If a compact set $\bar{U} \subset U$ and a number M exists such that, for every feasible solution (μ,u), there is a feasible solution $(\bar{\mu},\bar{u})$ satisfying: $\bar{\mu} \leqq M$, $\bar{u} \in \bar{U}$, $\bar{\mu} + \langle h,\bar{u}\rangle \leqq \mu + \langle h,u\rangle$; then problem (24.1) possesses an optimal solution.

<u>Proof:</u> In order to prove that the set of feasible solutions, denoted by $MU \subset R^1 \times U$, is closed, let $\{(\mu^i,u^i)\}_1^\infty \subset MU$ be a sequence of feasible solution which converges to a point $(\mu^0,u^0) \in R^1 \times U$. Suppose $(\mu^0,u^0) \notin MU$. Since $R^1_+ \times U_+$ is closed (U_+ by supposition) $\{(\mu^i,u^i)\}_1^\infty \subset MU$; $(\mu^i,u^i) \to (\mu^0,u^0)$, $i \to \infty$; $(\mu^0,u^0) \notin MU$ implies the existence of a $\tilde{w} \in W_+$ such that: $H[\tilde{w}],u^0\rangle + \mu^0 < r[\tilde{w}]$, $\langle H[\tilde{w}],u^i\rangle + \mu^i \geqq r[\tilde{w}]$ $i = 1,2,\ldots.$ Since by supposition $H[\tilde{w}] \subset U^*$, $|r[\tilde{w}]| < \infty$, this conflicts the presumption: $(\mu^i,u^i) \to (\mu^0,u^0)$, $i \to \infty$. In the context of this proof, this implies that the set of feasible solutions MU is closed.

The existence of a number M and a compact set $\bar{U}$, as mentioned above, implies that we may restrict ourselves to feasible solutions $(\mu,u) \in \underline{MU} := ([0,M]\times\bar{U}) \cap MU$. Since $([0,M]\times\bar{U})$ is compact ($\bar{U}$ by supposition), and since MU is closed, the set $\underline{MU}$ is compact. Then continuity of linear functional h implies (by the generalized Weierstrasz theorem ref. 3 , page 40) that programming problem $\{\sup \langle h,u\rangle+\mu \mid (\mu,u) \in \overline{MU}\}$ possesses an optimal solution and so, by the suppositions concerning number M and set $\bar{U}$, the programming problem (24.1) as well.

25. Applications to discrete-time convex infinite horizon programming problems.

Using definitions 2.4 to 2.6, and introducing slack-variables $\{y(t)\}_1^\infty \subset R_+^m$, the programming problem of §1 can be written:

$$\sup \sum_{t=1}^{\infty} \pi^t p[x(t);t] \;\left|\; \begin{array}{l} B[x(1);1]+y(1) = f(1)+A[x(0);0] \\ B[x(t+1);t+1]-A[x(t);t] + \\ \qquad + y(t+1) = f(t+1),\ t = 1,2,... \\ (x(t),y(t)) \in R_+^{n+m},\ t = 1,2,..., \end{array}\right. \tag{25.1}$$

or in the form is:

$$\sup_{x,y} q[x] \mid G[x]+y = g,\ x \in X_+,\ y \in Z_+, \tag{25.2}$$

where the spaces $X \subset 1^n$, $Z \subset 1^m$ will be specified later.

The dual problem (viz. §15, §24) gives rise to the problem.

$$\inf_{\mu,u} \mu+\langle g,u\rangle \;\left|\; \begin{array}{l} \langle u,G[x]\rangle+\mu \geqq q[x],\ \text{for all } x \in \tilde{X}_+ \\ (\mu,u) \in R_+^1 \times U_+, \end{array}\right. \tag{25.3}$$

U and $\tilde{X}$ being subspaces of l^m and l^n, later to be specified.

A straightforward calculations will shown that the restrictions of (25.3) can be written:

$$\left\{\sum_{t=1}^{\infty} u(t)'B[x(t);t]-u(t+1)'A[x(t);t]\right\}+\mu \geqq \sum_{t=1}^{\infty} \pi^t p[x(t);t]. \quad (25.4)$$

Since (viz. §1), $A[0;t] = 0$, $B[0;t] = 0$, $P[0;t]$, $t = 1,2,\ldots$, (25.4) implies: a point $(\mu,\{u(t)\}_1^{\infty}) \in R_+ \quad l_+^m$ satisfies (25.4) for all $\{x(t)\}_1^{\infty} \in l_+^n$ if, and only if, a sequence of numbers $\{\mu(t)\}_1^{\infty} \subset R_+^1$, $\sum_{t=1}^{\infty} \mu(t) = \mu$ exists such that, for all $z \in R_+^n$ and all periods $t = 1,2,\ldots$:

$$u(t)'B[z;t]-u(t+1)'A[z;t]+\mu(t) \geqq \pi^t p[z;t]. \quad (25.5)$$

In that way, the dual problem brings us to investigate the programming problem:

$$\left.\begin{array}{l} \inf \{u(1)'A[x(0);0] + \sum_{t=1}^{\infty} f(t)'u(t)+\mu(t)\}, \text{ subject to} \\ \left.\begin{array}{l} u(t)'B[z;t]-u(t+1)'A[z;t]+\mu(t)\geqq\pi^t p[z;t], \text{ for all } z\in R_+^n \\ \mu(t),u(t) \geqq 0 \end{array}\right\} t=1,2,\ldots \end{array}\right\}$$

(25.6)

Straightforward calculations will prove the following propositions:

<u>26. Proposition.</u>

Feasible solutions $\{x(t),y(t))\}_1^{\infty} \in l_+^{n+m}$, $\{(\mu(t),u(t))\}_1^{\infty} \in l_+^{1+m}$ of (25.1) and (25.6) satisfy:

$$\sum_{t=1}^{T} \pi^t p[x(t);t] = u(1)'A[x(0);0] + \sum_{t=1}^{T} \mu(t)+f(t)'u(t) -$$

$$- \sum_{t=1}^{T} u(t)'y(t) - \sum_{t=1}^{T} v[x(t);t]-u(T+1)'A[x(T);T] \text{ , } T = 1,2,\ldots,$$

where: $v[\cdot;t] := u(t)'B[\cdot;t]-u(t+1)'A[\cdot;t]+\mu(t)-\pi^t p[\cdot;t]$.

27. Proposition.

If, for every $z \in R_+^n$ and every period t: $A[z;t] \geqq 0$, then feasible solutions $\{(x(t),y(t)\}_1^\infty \in l_+^{n+m}$, $\{(\mu(t),u(t)\}_1^\infty \in l_+^{1+m}$ of (25.1) and (25.6) satisfy:

$$\sum_{t=1}^{T} \pi^t p[x(t);t] \leqq u(1)'A[x(0);0] + \sum_{t=1}^{T} \mu(t)+f(t)'u(t) \text{, } T = 1,2,\ldots$$

28. Proposition: a necessary condition for superiority.

If the problems (25.1) and (25.6) satisfy the conditions:

(a) For every $z \in R_+^n$ and every period t: $A[z;t] \geqq 0$.

(b) A sequence $\{(\bar{\mu}(t),\bar{u}(t)\}_1^\infty \in l_{\infty;1/\pi+}^{1+m}$ (def. 2.3) exists such that for some $\bar{v} \in \text{int}(R_+^n)$:

$$\bar{u}(t)'B[z;t]-\bar{u}(t+1)'A[z;t]+\bar{\mu}(t)-\pi^t p[z;t] \geqq \pi^t \bar{v}'z, \qquad (28.1)$$

for all $z \in R_+^n$ and all $t = 1,2,\ldots$.

(c) A sequence $\{\bar{x}(t)\}_1^\infty \in l_{\infty;1+}^n$ exists such that, for some $\bar{y} \in \text{int}(R_+^m)$:

$$\left.\begin{aligned}&B[\bar{x}(1);1] + \bar{y} \leqq f(1) + A[x(0);0]\\&B[\bar{x}(t+1);t+1] - A[\bar{x}(t);t] + \bar{y} \leqq f(t+1),\ t = 1,2,\ldots\end{aligned}\right\}\ (28.2);$$

Then:

- For every feasible solution $\{(x(t),y(t))\}_1^\infty \notin l_{1;\pi}^{n+m}$ of (25.1), there is an $\varepsilon > 0$ and a period S such that:

$$\sum_{t=1}^{T} \pi^t p[x(t);t] \leqq -\varepsilon + \sum_{t=1}^{T} \pi^t p[\bar{x}(t);t],\ T = S, S+1,\ldots$$

$\{\bar{x}(t)\}_1^\infty \in l_{\infty+}^n$ being the feasible solution of 28-c.

- For every feasible solution $\{(\mu(t),u(t))\}_1^\infty \notin l_{1;1+}^{1+m}$ of (25.6), there is an $\varepsilon > 0$ and a period S such that:

$$u(1)'A[x(0);0] + \sum_{t=1}^{T} \mu(t)+f(t)'u(t) \geqq \varepsilon + \bar{u}(1)'A[x(0);0] + $$
$$+ \sum_{t=1}^{T} \bar{\mu}(t)+f(t)'\bar{u}(t),\ T = 1,2,\ldots$$

$\{(\bar{\mu}(t),\bar{u}(t))\}_1^\infty \in l_{\infty;1/\pi+}^{1+m}$ being the feasible solution of (28-b).

29. Restatement of the problem.

In the next theorem we assume:

a) A number K exists such that for all x, y $\in R_+^n$:

$$p[x;t] - p[y;t] \mid \leqq K \|x-y\|,\ t = 1,2,\ldots$$

b) Numbers L_1, L_2 exist such that for all x, y $\in R_+^n$:

$$\left.\begin{aligned}&\|A[x;t] - A[y;t]\| \leqq L_1 \|x-y\|\\&\|B[x;t] - B[y;t]\| \leqq L_2 \|x-y\|\end{aligned}\right\}\ t = 1,2,\ldots$$

These conditions imply that for every $\{x(t)\}_0^\infty \in l^n_{1;\pi+}$:
$\{B[x(t+1);t+1)] - A[x(t);t]\}_0^\infty \in l^m_{1;\pi}$ and, in addition:
the convergency of the series $\{\sum_{t=1}^{T} \pi^t p[x(t);t]\}_{T=1}^\infty$.

In connection with the necessary conditions for superiority given in 28, this means that the original programming problems (25.1) and (25.6) can be replaced by

$$\sup_{x,y} q[x] \,|\, G[x]+y = g,\ x \in l^n_{1;\pi+},\ y \in l^m_{1;\pi+} \tag{29.1}$$

and

$$\inf_{\mu,u} \mu+\langle g,u\rangle \left| \begin{array}{l} \langle u,G[x]\rangle+\mu \geqq q[x],\ \text{for all } x \in l^n_{\infty;1+} \\ u \in l^m_{1;1+},\ \mu \in R^1_+ \end{array} \right. \tag{29.2}$$

In order to prove the existence of optimal solutions we assume:

c) Numbers M_1, M_2, and $\alpha \in]\pi,1[$ exist such that, for every $x \in l^n_{1;\pi+}$, $z \in l^m_{1;\alpha+}$ satisfying $G[x] \leqq g+z$, an $\bar{x} \in l^n_{1;\alpha+}$ exists for which: $\|\bar{x}\|_{1;\alpha} \leqq M_1+M_2 \|z\|_{1;\alpha}$, $G[\bar{x}] \leqq g+z$, and in addition: $q[\bar{x}] \geqq q[x]$.

d) Numbers N, and $\beta \in]\pi,1[$ exist such that, for every feasible $(\mu,u) \in R \times l^m_{1;1}$ of (29.2) a feasible solution $(\bar{\mu},\bar{u}) \in R_+$ $(\bar{\mu},\bar{u}) \in R^1_+ \times l^m_{1;1/\beta}$ such that $\bar{\mu}+\|\bar{u}\|_{1;1/\beta} \leqq N$, $\bar{\mu}+\langle g,\bar{u}\rangle \leqq \mu+\langle g,u\rangle$.

(Note: for <u>linear</u> infinite horizon problems it can be shown that the conditions 28-a to c imply the existence of numbers as mentioned in 29-c, d, viz. ref. 1).

<u>30. Theorem: discrete-time infinite horizon duality relations.</u>

Consider the programming problems of (25.1) and (25.6) where we restrict ourselves to feasible solutions $\{(x(t),y(t))\}_1^\infty \in l^{n+m}_{;+}$ and $\{(\mu(t),u(t))\}_1^\infty \in l^{1+m}_{1;1+}$ resp.

Suppose that all conditions of §1, §28, and §29 are satisfied. Then:

a) The problems both possess an optimal solution.

b) The supremum in (25.1) is equal to the infimum in (25.6).

c) Feasible solutions $\{(x(t),\ y(t))\}_1^\infty \in l^{n+m}_{1;\pi}$, $\{(\mu(t),u(t))\}_1^\infty \in l^{1+m}_{1;1}$ of (25.1) and (25.6), both, are both optimal if, and only if, simultaneously:

$$\left.\begin{aligned} & u(t)'y(t) = 0 \\ & u(t)'B[x(t);t]-u(t+1)'A[x(t);t]+\mu(t)-\pi^t p[x(t);t] = 0 \end{aligned}\right\} t=1,2,\ldots \tag{30.1}$$

$$u(t+1)'A[x(t);t] \to 0,\ t \to \infty \tag{30.2}$$

<u>Proof:</u> Consider programming problem (25.2):

$$\sup_{x,y} q[x]\ |G[x]+y = g,\ x \in l^n_{1;\alpha+}\ ,\ y \in l^m_{1;\alpha+}, \tag{30.3}$$

where the spaces X and Z are specified by $X := l^n_{1;\alpha}$, $Z := l^m_{1;\alpha}$: $\alpha \in]\pi,1[$ being the number appearing in 29-c. Since $(l^n_{1;\alpha})^* = l^n_{\infty;1/\alpha}$, $(l^m_{1;\alpha})^* = l^m_{\infty;1/\alpha}$, the dual problem of (30.3) (in the sense of §15), takes the form:

$$\inf_{\mu,u^*}\ \mu+\langle g,u^*\rangle \left|\begin{aligned} & \langle u^*,G[x]\rangle+\mu \geqq q[x],\ \text{for all } x \in l^n_{1;\alpha} \\ & u^* \in l^m_{\infty;1/\alpha},\ \mu \in R^1_+ \end{aligned}\right. \tag{30.4}$$

In connection with supposition 29-a and 29-b, the definition of $q[\cdot]$ and of $G[\cdot]$ implies that $q[\cdot]$ and $G[\cdot]$ are weak* continuous. Since the closed unit sphere in $l^n_{1;\alpha}$ is weak* compact (Alaoglu's theorem), and since $l^n_{1;\alpha}$ is weak* closed the weak* continuity of $G[\cdot]$, $q[\cdot]$ and supposition 29-c imply (by virtue of 21):

(1) Problem (30.3) possesses an optimal solution.

(2) The infimum in (30.4) is equal to the supremum in (30.3).

Now, consider programming problem (29.2). Let $N > 0$, $\beta \in]\pi,1[$ be the numbers of supposition 29-d. It can be shown that $\beta \in]\pi,1[$ implies the compactness of the set $\tilde{U} := \{u \in l^m_{1;1/\beta} \mid \|u\|_{1;1/\beta} \leqq N\}$ is compact in $l^m_{1;1}$. By virtue of §24, this implies:

(3) Problem (29.2) possesses an optimal solution, and so problem (25.6), as well.

Between the programming problems (29.1), (29.2), (30.3), and (39.4) we have the following relations:

(4) infimum (29.2) $\leqq$ infimum (30.4). Motivation: $\alpha \in]\pi,1[$ implies every feasible solution of (30.4) is a feasible solution of (29.2)

(5) sup (30.3) $\leqq$ sup (29.1). Motivation: $\alpha \in]\pi,1[$ implies every feasible solution of (30.3) is a feasible solution of (29.1)

(6) inf (29.2) $\leqq$ sup (29.1). Motivation: (4), (2), and (5) imply: inf (29.2) $\leqq$ inf (30.4) = sup (30.3) $\leqq$ sup (29.1)

(7) inf (29.2) $\geqq$ sup (29.1). Motivation: proposition 27 (inequality 27.2).

Combining (6) and (7), we may conclude: inf (29.2) = sup (29.1), which proves the b-part of the theorem. The a-part is implied by (1) and (3). The c-part follows from the equality inf (29.2) = = sup (29.1), proposition 26, and proposition 27.

31. Example.

The meaning of condition (30.2) can be illustrated with the help of the following example, where the functions $B[\cdot;t]$, $A[\cdot;t]$ and $p[\cdot;t]$ are supposed to be linear and constant over the periods. The functions are represented by the matrices A, B and the vector p, defined by:

$$A := \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}, \quad B := \begin{bmatrix} 9 & 0 \\ 0 & 10 \end{bmatrix}, \quad p := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad f(t) := \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \quad t = 1,2,\ldots.$$

The discount factor $\pi := 0.8$ and the initial vector $x(0)' := (1,1)$. For this example, one may verify that:

$$\hat{x}(t) := \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \left(\frac{10}{9}\right)^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad t = 1,2,\ldots \tag{31.1}$$

$$\hat{u}(t) := (0.8)^t \begin{bmatrix} 0 \\ \frac{1}{6} \end{bmatrix}, \quad \hat{\mu}(t) := 0, \; t = 1,2,\ldots, \tag{31.2}$$

are optimal solutions of primal problem (25.1) and dual problem (25.6) resp.; the value of the objective functions are 4. It appears that

$$\tilde{u}(t) := (0.8)^t \begin{bmatrix} 0 \\ \frac{1}{6} \end{bmatrix} + (0.9)^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \tilde{\mu}(t) := 0, \; t = 1,2,\ldots \tag{31.3}$$

is a feasible solutions of dual problem (25.6), with the property that $\{\hat{x}(t)\}_1^\infty$ and $\{\tilde{u}(t)\}_1^\infty$ (defined by 31.1 and 31.3, resp.) satisfy (30.1). However:

$$\tilde{u}(T+1)'A\hat{x}(T) = 9 + (0.8)^T \frac{2}{3}, \quad T = 1,2,\ldots, \tag{31.4}$$

so that condition (30.2) is not satisfied. Since, for $\{\tilde{u}(t)\}_1^\infty$, the value of the objective function is 13, and since the infimum is 4, we find that $\{\tilde{u}(t)\}_1^\infty$ is not optimal.

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